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# WHITE NOISE ANALYSIS AND THE BOUNDARY VALUE PROBLEM IN THE SPACE OF STOCHASTIC DISTRIBUTIONS(Quantum Stochastic Analysis and Related Fields)

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# WHITE NOISE ANALYSIS AND THE BOUNDARY VALUE PROBLEM IN THE SPACE OF STOCHASTIC DISTRIBUTIONS

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## ABSTRACT

We introduce the concept of functional process and consider the stochastic boundary value problem and discuss the convergence of its asymptotic solution process. The formulation of the problem is totally based upon the white noise analysis. In particular the so-called Hermite transform does play an essential role in derivation of the corresponding partial differential equation. One of the peculiar features under adoption of HLOUZ formalism (1993) consists in interpretation of the stochastic integral term as an integral of the Wick product of white noise functionals. We regard the solution of the problem as a Kondratiev space valued functional process, and the corresponding asymptotic solution satisfies some stochastic partial differential equation with a martingale term.

## 1. Preliminaries

### 1.1 White Noise Probability Space

Let  $d \in \mathbb{N}$  fixed, and it indicates the parameter dimension.  $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$  denotes a Schwartz space on  $\mathbb{R}^d$ .  $\mathcal{S}$  is a Fréchet space under a family of seminorms  $\|\cdot\|_{k,\alpha}$ , where

$$\|f\|_{k,\alpha} = \sup_{x \in \mathbb{R}^d} (1 + |x|^k) |\partial^\alpha f(x)|, \quad k \geq 0,$$

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_d$ , and  $\partial^\alpha f = \partial^{|\alpha|} f / \partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \dots \partial^{\alpha_d} x_d$ .  $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$  is a dual of  $\mathcal{S}$ , equipped with weak-\* topology. It is called the space of tempered distributions. We denote by  $\mathcal{B} = \mathcal{B}(\mathcal{S}')$  the family of Borel subsets of  $\mathcal{S}'$ . By the Bochner–Minlos theorem, there exists a unique Gaussian probability measure (called a white noise measure) on  $\mathcal{B}$  such that

$$\int_{\mathcal{S}'} e^{i\langle x, \varphi \rangle} d\mu(x) = e^{-\frac{1}{2}|\varphi|_2^2}, \quad \forall \varphi \in \mathcal{S},$$

where  $|\cdot|_2$  is a  $L^2(\mathbb{R}^d)$ -norm. We call the triplet  $(\mathcal{S}', \mathcal{B}, \mu)$  a white noise probability space. The canonical bilinear form  $\langle x, \varphi \rangle$ , for  $x \in \mathcal{S}$ ,  $\varphi \in L^2(\mathbb{R}^d)$  is defined as follows: for  $\forall \varphi \in L^2(\mathbb{R}^d)$ ;  $\exists \{\varphi_k\} \subset \mathcal{S}$  such that  $\varphi_k \rightarrow \varphi$  in  $L^2(\mathbb{R}^d)$  as  $k$  approaches to infinity, and define  $\langle x, \varphi \rangle := L^2\text{-}\lim_{k \rightarrow \infty} \langle x, \varphi_k \rangle$ . In particular, when we define

$$\tilde{B}_t(x) := \langle x, \chi_{[0,t_1] \times \dots \times [0,t_d]} \rangle, \quad \text{for } t_k \geq 0, \quad t = (t_1, \dots, t_d),$$

then it is well-known that there exists a  $t$ -continuous version  $B_t$  of  $\tilde{B}_t$ , and we call it a  $d$ -parameter Brownian motion, where  $\chi_A$  denotes an indicator of the set  $A$ . Next we introduce a  $d$ -parameter white noise process (WN process for short)  $W \equiv W_\varphi$ , which can be expressed in terms of Itô integral with respect to  $d$ -parameter Brownian motion  $B = (B_t(x))$ ,  $t \in \mathbb{R}^d$ ; i.e., the white noise process is a mapping  $W: \mathcal{S} \times \mathcal{S}' \rightarrow \mathbb{R}$ , given by

$$(1) \quad W(\varphi, x) = W_\varphi(x) = \langle x, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(t) dB_t(x), \quad x \in \mathcal{S}', \quad \varphi \in \mathcal{S}.$$

### 1.2 The Space $(L^2)$ and its Representations

Let  $\tilde{L}^2$  be the totality of square integrable measurable functions on  $\mathcal{S}'$  with respect to the white noise measure  $\mu$ . We denote by the symbol  $(L^2) = L^2(\mathcal{S}', \mu)$  the quotient space of  $\tilde{L}^2$  by the equivalence class, namely, the equivalent relation  $f \sim g$  is given by  $\|f - g\|_2 = 0$ . The Wiener-Itô expansion theorem gives the following decomposition of the space  $(L^2)$ : indeed,  $(L^2) = L^2(\mathcal{S}', \mu) = \sum_{n=0}^{\infty} \bigoplus K_n$ , where each  $K_n$  is the totality of multiple Wiener integrals  $I_n(f_n)$  of order  $n$ , and  $f_n$  is an element of the symmetric  $L^2$ -space  $\hat{L}^2((\mathbb{R}^d)^n)$ . For  $\forall F \in (L^2)$  we have the expression:

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^{dn}} f_n(u) dB_u^{\otimes n}(x) \quad f_n \in \hat{L}^2((\mathbb{R}^d)^n) \\ &= \sum_{n=0}^{\infty} \int \cdots \int_{\mathbb{R}^{dn}} f_n(u_1, \dots, u_n) dB^{\otimes n}(u_1, \dots, u_n)(x), \quad u_k \in \mathbb{R}^d. \end{aligned}$$

For the norm  $\|\cdot\|$  (or  $\equiv \|\cdot\|_2$ ) of the Hilbert space  $(L^2)$ , we have

$$\|F\|^2 = \sum_{n=0}^{\infty} n! |f_n|_2^2,$$

for  $f_n \in \hat{L}^2((\mathbb{R}^d)^n)$ .

We consider an alternative representation of the element of  $(L^2)$ . Let  $h_n(y)$ ,  $n = 0, 1, 2, \dots$ , be Hermite polynomials defined by

$$h_n(y) := (-1)^n e^{\frac{y^2}{2}} \frac{d^n}{dy^n} (e^{-\frac{y^2}{2}}), \quad y \in \mathbb{R}.$$

Then it is well-known that the Hermite functions  $\xi_n(y)$  are defined, by employing the Hermite polynomials, as

$$\xi_n(y) = \pi^{-\frac{1}{4}} \{(n-1)!\}^{-\frac{1}{2}} e^{-\frac{y^2}{2}} h_{n-1}(\sqrt{2}y), \quad n \geq 1.$$

Note that  $\{\xi_n(y)\}_{n=1}^\infty$  forms an orthonormal basis of  $L^2(\mathbb{R})$  for the case  $d = 1$ . Let  $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{Z}_+^d$  be a multi-index. Then there is always a proper ordering so that we may rearrange the elements numerically and make it countable in the following manner:

$$\{\beta = (\beta_1, \dots, \beta_d)\} = \{\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \dots\}, \quad \text{and} \quad \beta^{(n)} = (\beta_1^{(n)}, \beta_2^{(n)}, \dots, \beta_d^{(n)}).$$

Therefore we can define  $e_n \equiv e_{\beta^{(n)}} := \xi_{\beta_1^{(n)}} \otimes \xi_{\beta_2^{(n)}} \otimes \dots \otimes \xi_{\beta_d^{(n)}}$ . Note that  $e_k \in \mathcal{S}(\mathbb{R}^d)$  for each  $k$ . Thus we obtain an orthonormal basis  $\{e_n\}_n = \{e_1, e_2, e_3, \dots\} (\subset \mathcal{S})$  for  $L^2(\mathbb{R}^d)$ . Set

$$\theta_j(x) := W_{e_j}(x) = \int_{\mathbb{R}^d} e_j(t) dB_t(x) = \langle x, e_j \rangle, \quad \text{for } j = 1, 2, \dots$$

For every multi-index  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$ , we define  $h_\alpha(u_1, \dots, u_m) := h_{\alpha_1}(u_1) \cdot h_{\alpha_2}(u_2) \cdot \dots \cdot h_{\alpha_m}(u_m)$ , and set

$$H_\alpha(x) := h_\alpha(\theta_1(x), \dots, \theta_m(x)) = \prod_{j=1}^m h_{\alpha_j}(\theta_j(x)) = \prod_{j=1}^m h_{\alpha_j}(\langle x, e_j \rangle).$$

It hence follows that with  $|\alpha| = n = \alpha_1 + \dots + \alpha_m$ ,

$$\begin{aligned} (2) \quad \int_{(\mathbb{R}^d)^n} e^{\otimes \alpha} dB^{\otimes |\alpha|} &\equiv \int_{(\mathbb{R}^d)^n} e_1^{\hat{\otimes} \alpha_1} \hat{\otimes} \dots \hat{\otimes} e_m^{\hat{\otimes} \alpha_m} dB_t^{\hat{\otimes} n} \quad (t \in \mathbb{R}^d) \\ &= \prod_{j=1}^m h_{\alpha_j}(\theta_j) = H_\alpha(x). \end{aligned}$$

**Theorem 1.** (i)  $\{H_\alpha(\cdot); \alpha \in \mathbb{N}^m : m = 0, 1, 2, \dots\}$  forms an orthonormal basis of the Hilbert space  $(L^2)$ .

(ii)  $\mathbb{E}[H_\alpha^2] = \|H_\alpha\|^2 = \alpha!$ , where  $\alpha! = \prod_{j=1}^m \alpha_j!$ ,  $\alpha = (\alpha_1, \dots, \alpha_m)$ .

On this account, an arbitrary element  $F$  of  $(L^2)$  can be expressed as

$$(3) \quad F(x) = \sum_{\alpha} c_{\alpha} \cdot H_{\alpha}(x), \quad c_{\alpha} \in \mathbb{R}, \quad \alpha \in \mathbb{Z}^m, \quad \forall m.$$

Moreover, the equality  $\|F\|^2 = \sum_{\alpha} \alpha! c_{\alpha}^2$  holds.

*Example 1.* (White Noise Process) Recall the white noise process  $W_{\psi}$  (cf. Eq.(1)), which was introduced in the end of the section 1.1. For  $\psi \in \mathcal{S}$ ,  $x \in \mathcal{S}'$ ,

$$W_{\psi}(x) = \langle x, \psi \rangle = \int_{\mathbb{R}^d} \psi(t) dB_t(x) \equiv \int \dots \int_{\mathbb{R}^d} \psi(t_1, \dots, t_d) dB_{t_1 \dots t_d}(x).$$

Since we have  $\psi(t) = \sum_{k=1}^{\infty} (\psi, e_k) e_k \in \mathcal{S}$  by making use of the orthonormal basis  $\{e_k\}$  for  $L^2(\mathbb{R}^d)$ , it is easy to see that

$$(4) \quad W_{\psi}(x) = \sum_{k=1}^{\infty} (\psi, e_k) \int_{(\mathbb{R}^d) \times 1} e^{\hat{\otimes} \varepsilon(k)} dB^{\otimes |\varepsilon(k)|} = \sum_{k=1}^{\infty} (\psi, e_k) H_{\varepsilon(k)}(x),$$

where we used Eq.(2) and  $\alpha = \varepsilon_k = \varepsilon(k) = (0, \dots, 0, \overset{k}{1}, 0, \dots, 0) \in \mathbb{Z}_+^m$ .

### 1.3 Stochastic Distributions

Recall that we have a Gelfand triple:  $\mathcal{S} \subset L^2(\mathbb{R}^d) \subset \mathcal{S}'$ . It is possible to construct a similar structure in functional level (i.e. infinite dimensional case), which is modelled on the above-mentioned Gelfand triple in function level (i.e. finite dimensional case). Actually the second quantized operator  $\Gamma(A)$  plays an essential role in its construction (see e.g. [HKPS]), where  $A$  is a positive selfadjoint operator in  $L^2(\mathbb{R}^d)$  with Hilbert-Schmidt inverse. The standard construction (cf. pp.33-35,[OB] or [D5]) gives a Gelfand triple  $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$ , where  $(\mathcal{S})$  is the space of test white noise functionals and  $(\mathcal{S})^*$  is the space of generalized white noise functionals. And besides the latter may be called the space of Hida distributions. The Potthoff-Streit characterization theorem (cf. pp.123-134, [HKPS]) for those spaces are based on the  $S$ -transform in white noise calculus. In line with this characterization, a generalization of Hida distributions has been established ([OB],[D7]). However, in fact there is another characterization based on the so-called chaos expansion of functionals, whose basic concept is nothing but the alternative representation given by Eq.(3) in the previous section. For near-future application's sake, we will go to the other way, different from the standard setting in white noise analysis. For  $(L^2) \ni F$ , we have the chaos expansion  $F(x) = \sum_{\alpha} c_{\alpha} H_{\alpha}(x)$ . We are now in a position to state the characterization of the white noise test functionals and Hida distributions in terms of the coefficients of their Hermite transforms (see the next section) due to Zhang [Z].

**Theorem 2.** (i)  $F \in (\mathcal{S})$  if and only if the condition

$$\sup_{\alpha} c_{\alpha}^2 \cdot \alpha! (2\mathbb{N})^{\alpha k} < \infty$$

holds for any  $k < \infty$ ,  $k \in \mathbb{N}$ , where  $(2\mathbb{N})^{\alpha} := \prod_{j=1}^m (2^d \cdot \beta_1^{(j)} \beta_2^{(j)} \dots \beta_d^{(j)})^{\alpha(j)}$  if  $\alpha = (\alpha_1, \dots, \alpha_m)$  with  $\alpha_j = \alpha(j)$  for simplicity.

(ii)  $G \in (\mathcal{S})^*$ ,  $G = \sum_{\alpha} b_{\alpha} H_{\alpha}$  (formal series) if and only if the condition

$$\sup_{\alpha} b_{\alpha}^2 \cdot \alpha! (2\mathbb{N})^{-\alpha q} < \infty$$

holds for some  $q > 0$ .

It is interesting to note that the action of  $G$  on  $F$  is given by

$$(5) \quad \langle G, F \rangle = \sum_{\alpha} \alpha! b_{\alpha} \cdot c_{\alpha}$$

if  $G \in (\mathcal{S})^*$  such that  $G = \sum_{\alpha} b_{\alpha} H_{\alpha}$  and  $F \in (\mathcal{S})$  such that  $F = \sum_{\alpha} c_{\alpha} H_{\alpha}$ .

Next we shall introduce the Kondratiev spaces [KSW].

**Definition 1.** (a) Let  $0 \leq \rho \leq 1$ . We say  $f \in (\mathcal{S})^\rho$  if  $f = \sum_\alpha c_\alpha \cdot H_\alpha \in (L^2)$  such that

$$(6) \quad \|f\|_{\rho,k}^2 := \sum_\alpha c_\alpha^2 \cdot (\alpha)^{1+\rho} (2\mathbb{N})^{\alpha k} < \infty, \quad (\forall k < \infty).$$

We call this  $(\mathcal{S})^\rho$  the Kondratiev space of stochastic test functions.

(b) Let  $0 \leq \rho \leq 1$ . We say  $F \in (\mathcal{S})^{-\rho}$  if  $F = \sum_\alpha b_\alpha \cdot H_\alpha$  such that

$$(7) \quad \sum_\alpha b_\alpha^2 \cdot (\alpha!)^{1-\rho} (2\mathbb{N})^{-\alpha q} < \infty, \quad (\exists q < \infty),$$

where  $q$  need to be large enough (i.e.  $q \gg 1$ ).  $(\mathcal{S})^{-\rho}$  is called the Kondratiev space of stochastic distributions.

The family of seminorms  $\|f\|_{\rho,k}^2$  ( $k = 1, 2, \dots$ ) gives rise to a topology on the space  $(\mathcal{S})^\rho$ . In fact, the space  $(\mathcal{S})^{-\rho}$  can be regarded as a dual of  $(\mathcal{S})^\rho$  by the action  $\langle F, f \rangle = \sum_\alpha b_\alpha c_\alpha \cdot \alpha!$  if  $F = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^{-\rho}$  and  $f = \sum_\alpha c_\alpha H_\alpha \in (\mathcal{S})^\rho$ . It follows therefore that

$$(8) \quad (\mathcal{S})^1 \subset (\mathcal{S})^\rho \subset (\mathcal{S})^0 = (\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^* = (\mathcal{S})^{-0} \subset (\mathcal{S})^{-\rho} \subset (\mathcal{S})^{-1}.$$

## 2. Elementary Wick Calculus

### 2.1 Wick Product $\diamond$

The purpose of this section consists in definition of the Wick product and its extension for application to stochastic equations. We shall introduce first the primitive definition of the Wick product, and later on try to extend it to the largest space, namely the Kondratiev space.

*N.B.* We already know that there exist much larger spaces of generalized functionals in white noise calculus, such as the Meyer-Yan space  $\mathcal{M}^*$  (cf. *LNM* **1485** (1991)), and the Carmona-Yan space  $\tilde{\mathcal{M}}^*$  (cf. *Prog. Probab.* **36** (1995)). We have the following inclusion:

$$(L^2) \subset (\mathcal{S})^* \subset (\mathcal{S})^{-\beta} \subset \mathcal{M}^* \subset \tilde{\mathcal{M}}^*.$$

Moreover there are continuous embeddings:  $\tilde{\mathcal{M}} \hookrightarrow \mathcal{M} \hookrightarrow (L^2) \hookrightarrow \mathcal{M}^* \hookrightarrow \tilde{\mathcal{M}}^*$ . In addition,  $\tilde{\mathcal{M}}$  is a nuclear Fréchet space which is stable under Wick and Wiener products. While,  $\tilde{\mathcal{M}}^*$  is the topological dual of the locally convex topological vector space  $\tilde{\mathcal{M}}$ . However, we need not use those spaces in this paper. The Kondratiev space is large enough to discuss the stochastic problem here in question.

In accordance with [HLOUZ1], [HLOUZ2], we define the Wick product of  $X$  and  $Y$  as

$$(9) \quad X \diamond Y := \iint_{(\mathbb{R}^d)^2} \varphi \otimes \psi dB^{\otimes 2}$$

if  $X = \langle x, \varphi \rangle = \int_{\mathbb{R}^d} \varphi dB$ , (for  $x \in \mathcal{S}'$ ,  $\varphi \in \mathcal{S}$ ) and  $Y = \langle x, \psi \rangle = \int_{\mathbb{R}^d} \psi dB$ , (for  $x \in \mathcal{S}'$ ,  $\psi \in \mathcal{S}$ ). We can extend it with ease to  $(L^2)$  by making use of the expression:

$$(L^2) \ni F(x) = \sum_{n=0}^{\infty} \int \cdots \int_{(\mathbb{R}^d)^n} f_n(u_1, \dots, u_n) dB_u^{\otimes n}, \quad (f_n \in \hat{L}^2(\mathbb{R}^{dn})).$$

**Definition 2.** (*Representation by Expansion*) If  $X$  and  $Y$  are elements of  $(L^2)$  such that  $X = \sum_{n=0}^{\infty} \int_{(\mathbb{R}^d)^n} f_n dB^{\otimes n}$  and  $Y = \sum_{m=0}^{\infty} \int_{(\mathbb{R}^d)^m} g_m dB^{\otimes m}$ , then the Wick product of  $X$  and  $Y$  is defined by

$$X \diamond Y = \sum_{n,m=0}^{\infty} \int \cdots \int_{(\mathbb{R}^d)^{n+m}} f_n \otimes g_m dB^{\otimes(n+m)},$$

where the right hand side is considered as convergence in  $L^1(\mathcal{S}', \mu)$ .

Next let us consider the alternative definition corresponding to the representation Eq.(3).

**Definition 3.** If  $X$  and  $Y$  are elements of  $(L^2)$  such that  $X = \sum_{\alpha} a_{\alpha} H_{\alpha}$ , and  $Y = \sum_{\beta} b_{\beta} H_{\beta}$ , then

$$X \diamond Y = \sum_{\alpha, \beta} a_{\alpha} b_{\beta} \cdot H_{\alpha+\beta},$$

where we consider the right hand side as convergence in  $L^1(\mathcal{S}', \mu)$  as far as it exists.

Needless to say, the above two definitions are equivalent. A direct computation leads to the equivalence. As a matter of fact, by taking Eq.(2) into account we can easily get

$$\begin{aligned} H_{\alpha} \diamond H_{\beta} &= \left( \prod_{j=1}^m h_{\alpha_j}(\theta_j) \right) \diamond \left( \prod_{i=1}^k h_{\beta_i}(\theta_i) \right) = \left( \int_{(\mathbb{R}^d)^n} e^{\otimes \alpha} dB^{\otimes |\alpha|} \right) \left( \int_{(\mathbb{R}^d)^l} e^{\otimes \beta} dB^{\otimes |\beta|} \right) \\ &= \int_{(\mathbb{R}^d)^{|\alpha+\beta|}} e^{(\alpha+\beta)} dB^{\otimes |\alpha+\beta|} = H_{\alpha+\beta}(x), \end{aligned}$$

with  $n = |\alpha| = \alpha_1 + \cdots + \alpha_m$  and  $l = |\beta| = \beta_1 + \cdots + \beta_k$ . Note that the Wick product  $X \diamond Y \equiv \sum_{\alpha, \beta} a_{\alpha} b_{\beta} \cdot H_{\alpha+\beta}$  which we have defined is independent of the choice of the base  $\{e_k\}$  of  $L^2(\mathbb{R}^d)$ .

*Example 2.* (Wick Product and Stochastic Integral: cf. p.398, [HLOUZ1]) If  $Y_t$  is an adapted bounded stochastic process defined on the white noise probability space  $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{S}', \mathcal{B}, \mu)$ , then we have the following equality:

$$(10) \quad \int_0^T Y_t(x) dB_t(x) = \int_0^T Y_t \diamond W_t(x) dt.$$

## 2.2 Wick Product of Distributions and Wick Exponential

Likewise, we can define the Wick product even for Hida distributions. In general, the spaces of stochastic distributions are stable under the Wick product. However, some smaller spaces are not always stable. Actually the followings are verified:

- (a) If  $F = \sum_{\alpha} a_{\alpha} H_{\alpha} \in (\mathcal{S})^*$ , and if  $G = \sum_{\beta} b_{\beta} H_{\beta} \in (\mathcal{S})^*$ , then  $F \diamond G = \sum_{\alpha, \beta} a_{\alpha} b_{\beta} \cdot H_{\alpha+\beta}$  holds.
- (b) If  $f, g \in (\mathcal{S})$ , then  $f \diamond g \in (\mathcal{S})$ .
- (c) However, for  $F, G \in (L^2)$ ,  $F \diamond G \notin (L^2)$  (not always!).
- (d) For  $X, Y \in L^1(\mathcal{S}', \mu)$ , suppose that there are  $X_n, Y_n \in (L^2)$  such that  $X_n \rightarrow X$  in  $L^1(\mathcal{S}', \mu)$ , and  $Y_n \rightarrow Y$  in  $L^1(\mathcal{S}', \mu)$  (as  $n \rightarrow \infty$ ). If  $\exists Z := \lim_{n \rightarrow \infty} X_n \diamond Y_n \in L^1(\mathcal{S}', \mu)$ , then we define  $X \diamond Y = Z$ .

It is interesting to note that the discussion in  $L^1(\mathcal{S}', \mu)$  is very delicate, because the space  $L^1(\mathcal{S}', \mu)$  is not necessarily contained in the space  $(\mathcal{S})^*$  of Hida distributions [HLOUZ1]. Next we shall introduce the Wick exponential, which is one of the most important tools in Wick calculus applied to stochastic differential equations in the standpoint of how to solve the problem. If  $X$  belongs to  $L^1(\mathcal{S}', \mu)$ , then we define the Wick exponential

$$(11) \quad \text{Exp} X := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}.$$

Of course, this definition is well-defined if there exists the Wick powers of  $X$ , namely,  $\exists X^{\diamond n}$  for any  $n$ , and if the series is convergent in  $L^1(\mathcal{S}', \mu)$ . Furthermore, we obtain the exponential rule:

$$(12) \quad \text{Exp}(X + Y) = \text{Exp} X \diamond \text{Exp} Y.$$

*Example 3.* ( $\text{Exp} W_{\psi}$ : the Wick exponential of WN process) Since we have  $\sum_{n=0}^{\infty} h_n(x) t^n / n! = \exp\{tx - t^2/2\}$ , it is easy to see that the WN process satisfies the relation

$$\text{Exp} W_{\psi} = \exp\left(W_{\psi} - \frac{1}{2} |\psi|_2^2\right).$$



Let  $\mathcal{A}$  be the algebra generated by  $\exp(W_\psi)$ . Since  $\mathcal{A}$  is dense in  $(\mathcal{S})$ , immediately  $\text{Exp}W_\psi \in (\mathcal{S})$ . Thus it follows that  $\text{Exp}W_\psi \in L^p(\mathcal{S}', \mu)$ , for any  $p \in [1, \infty)$ .

For the elements of the Kondratiev space, we define

$$(13) \quad F \diamond G := \sum_{\alpha, \beta} a_\alpha b_\beta \cdot H_{\alpha+\beta},$$

if  $F = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})^{-1}$  and  $G = \sum_\beta b_\beta H_\beta \in (\mathcal{S})^{-1}$ . The well-definedness above is guaranteed by the following lemma.

**Lemma 1.** (i)  $f, g \in (\mathcal{S})^1$ , then  $f \diamond g \in (\mathcal{S})^1$ .  
(ii)  $F, G \in (\mathcal{S})^{-1}$ , then  $F \diamond G \in (\mathcal{S})^{-1}$ .

### 2.3 Hermite Transform

We shall introduce the Hermite transform, which is a powerful tool in white noise calculus, especially when it is used for the study of stochastic differential equations.

**Definition 4.** (Hermite Transform  $\mathcal{H}$ ) For  $\forall F \in (L^2)$  (resp.  $(\mathcal{S})^*$ ,  $(\mathcal{S})^{-1}$ ) such that  $\exists$  its chaos expansion  $F = \sum_\alpha c_\alpha H_\alpha$ , the Hermite transform  $\mathcal{H}$  of  $F$  is defined respectively as

$$(14) \quad \mathcal{H}F \equiv \tilde{F} := \sum_\alpha c_\alpha z^\alpha,$$

where  $z = (z_1, z_2, \dots) \in \mathbb{C}^{\mathbb{N}}$ .

Note that, in the above, if  $\alpha = (\alpha_1, \dots, \alpha_m)$  then  $z^\alpha = z_1^{\alpha_1} \dots z_m^{\alpha_m}$ .

**Proposition 3 [LOU].** (i) If  $X = \sum_\alpha c_\alpha H_\alpha \in (L^2)$ , then for each  $M (< \infty)$ , each  $n \in \mathbb{N}$ , its Hermite transform  $\tilde{X}(z) = \sum_\alpha c_\alpha z^\alpha$  converges absolutely for  $z = (z_1, z_2, \dots, z_n, 0, 0, \dots, 0)$ ,  $|z_k| \leq M$  ( $\forall k$ ).

(ii) (Therefore) for each  $n$ ,

$$\tilde{X}^{(n)}(z_1, \dots, z_n) \equiv \tilde{X}(z_1, \dots, z_n, 0, \dots, 0)$$

is analytic on  $\mathbb{C}^n$ .

**Theorem 4 [LOU].** Suppose that  $X, Y \in (L^2)$  satisfying  $X \diamond Y \in (L^2)$ . Then

$$\mathcal{H}(X \diamond Y) = \mathcal{H}(X) \cdot \mathcal{H}(Y)$$

holds, where “.” indicates the usual complex product.

*Example 4.* (a) (WN process  $W_\varphi$ ) Recall that  $W_\varphi(x) = \sum_k (\varphi, e_k) H_{\varepsilon(k)}(x) = \sum_k (\varphi, e_k) h_1(\theta_k)$  for  $x \in \mathcal{S}'$ ,  $\varphi \in \mathcal{S}$  (see Example 1). Then we have

$$\mathcal{H}(W_\varphi) = \tilde{W}_\varphi(z) = \sum_{k=1}^{\infty} (\varphi, e_k) \cdot z_k.$$

(b) (The Square of WN process:  $W_\varphi^{\diamond 2} = W_\varphi \diamond W_\varphi$ ) We have

$$\mathcal{H}(W_\varphi^{\diamond 2}) = \sum_{k,j=1}^{\infty} (\varphi, e_k)(\varphi, e_j) z_k \cdot z_j.$$

For Hida distributions, the same assertion as Theorem 4 holds; indeed, for  $F, G \in (\mathcal{S})^*$ ,  $\mathcal{H}(F \diamond G) = \mathcal{H}F \cdot \mathcal{H}G$ . What about the Kondratiev space? Is the same assertion valid for the elements of  $(\mathcal{S})^{-\rho}$ ?

*Remark 1.* If  $F$  lies in  $(\mathcal{S})^{-\rho}$  for  $\rho < 1$ , then it is easy to see that  $\mathcal{H}F(z_1, z_2, \dots)$  converges for any finite sequence  $Z = (z_1, z_2, \dots, z_m)$  of complex numbers for each  $m \in \mathbb{N}$ .

*Remark 2.* If  $F$  is an element of  $(\mathcal{S})^{-1}$ , then we can only obtain the convergence of  $\mathcal{H}F(z_1, z_2, \dots)$  in a neighborhood of the origin. Actually we have  $\mathcal{H} = \tilde{F} = \sum_\alpha c_\alpha \cdot z^\alpha$  for  $F = \sum_\alpha c_\alpha H_\alpha$ . So that, we get

$$(15) \quad \sum_\alpha |c_\alpha| \cdot |z^\alpha| \leq \left\{ \sum_\alpha c_\alpha^2 \cdot (2\mathbb{N})^{-\alpha q} \right\}^{1/2} \cdot \left\{ \sum_\alpha |z^\alpha|^2 \cdot (2\mathbb{N})^{\alpha q} \right\}^{1/2}.$$

The first term of the right hand side in Eq.(15) clearly converges for  $q \gg 1$  (large enough), because  $F \in (\mathcal{S})^{-1}$ . For such a value of  $q (\gg 1)$ , the second factor is convergent if  $z$  is taken from the set

$$(16) \quad \mathbb{B}_q(\delta) := \left\{ \zeta = (\zeta_1, \zeta_2, \dots) \in \mathbb{C}^{\mathbb{N}}; \sum_{\alpha \neq 0} |\zeta^\alpha|^2 \cdot (2\mathbb{N})^{\alpha q} < \delta^2 \right\}$$

for some  $\delta < \infty$  (cf. [HLOUZ2]).

**Proposition 5.** *If  $F, G \in (\mathcal{S})^{-1}$ , then*

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)$$

*holds for any  $z \in \mathbb{C}^{\mathbb{N}}$  so that both  $\mathcal{H}F$  and  $\mathcal{H}G$  may exist.*

The next assertion is of importance in applicational basis, especially when we apply the Hermite transform to rewrite the stochastic equation into an ordinary one and discuss the convergence of its approximate solutions. The topology on  $(\mathcal{S})^1$  can conveniently be expressed in terms of Hermite transforms as follows.

**Proposition 6.** *The following two convergences are equivalent:*

- (i)  $X_n \rightarrow X$  in  $(\mathcal{S})^{-1}$ .
- (ii)  $\exists \delta > 0, q < \infty, M < \infty$  such that

$$\mathcal{H}X_n(z) \rightarrow \mathcal{H}X(z) \quad (\text{as } n \rightarrow \infty) \quad \text{for } z \in \mathbb{B}_q(\delta)$$

and  $|\mathcal{H}X_n(z)| \leq M$  for all  $n = 1, 2, \dots, \forall z \in \mathbb{B}_q(\delta)$ .

**Theorem 7.** *(Characterization for the Kondratiev Space) Suppose that  $g(z_1, z_2, \dots)$  be a bounded analytic function on  $\mathbb{B}_q(\delta)$  ( $\exists \delta > 0, q < \infty$ ). Then there exists an element  $X$  in  $(\mathcal{S})^{-1}$  such that  $\mathcal{H}X = g$  holds.*

**Corollary 8.** *Suppose that  $g = \mathcal{H}X$  ( $\exists X \in (\mathcal{S})^{-1}$ ). Let  $f$  be an analytic function in the neighborhood of  $g(0)$  in  $\mathbb{C}$ . Then there exists an element  $Y$  in  $(\mathcal{S})^{-1}$  such that  $\mathcal{H}Y = f \circ g$ .*

*Example 5.* Let  $X \in (\mathcal{S})^{-1}$ . Then  $X \diamond X = X^{\diamond 2} \in (\mathcal{S})^{-1}$  is always true by (ii) of Lemma 1. More generally,  $X^{\diamond n} \in (\mathcal{S})^{-1}$  holds for  $\forall n \in \mathbb{N}$ . Hence we attain that

$$\text{Exp}X \equiv \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n} \in (\mathcal{S})^{-1}$$

by applying Corollary 8 with  $f(z) = \exp(z)$ .

*Remark 3.* The Hermite transform  $\mathcal{H}$  and the S-transform in white noise analysis are closely connected. As a matter of fact, the following relation holds.

$$\mathcal{H}F(z_1, z_2, \dots, z_m) = SF(z_1 e_1 + z_2 e_2 + \dots + z_m e_m)$$

for any  $z = (z_1, z_2, \dots, z_m) \in \mathbb{C}^m, (\exists m \in \mathbb{N})$ .

**Theorem 9.** *(Interchangeability of Integration and Wick Product) Assume that  $F(\cdot, \cdot) \in L^2(\mathcal{S}' \times \mathcal{S}', \mu \otimes \mu)$ . For any  $G \in (\mathcal{S})^*$ ,*

$$\int_{\mathcal{S}'} F(\eta, x) \diamond G(x) d\mu(\eta) = \int_{\mathcal{S}'} F(\eta, x) d\mu(\eta) \diamond G(x).$$

**Theorem 10.** *Assume that  $Y \in (L^2)$ , and  $\psi \in C_0^\infty(\mathbb{R})$  such that  $\text{supp } \psi \subset [a, b]$ . If  $\psi(s)Y(\omega)$  is Skorohod integrable, then*

$$Y(\omega) \diamond W_\psi(\omega) = \int_a^b \psi(s) \cdot Y(\omega) \delta B_s(\omega)$$

*holds, where the right hand side means the Hitsuda-Skorohod integral (cf. [HKPS]).*

### 3. Functional Process

#### 3.1 $(L^p)$ -Functional Process

We write  $L^p(\mathcal{S}', \mu)$  as  $(L^p)$ . When  $X$  is an  $(L^p)$ -functional process, we write  $X \in \mathcal{L}^p$ .

**Definition 5.** ( $(L^2)$ -Functional Process) We say  $X \in \mathcal{L}^2$  if  $X = X(\varphi, t, x)$  is a mapping  $: \mathcal{S} \times \mathbb{R}^d \times \mathcal{S}' \rightarrow \mathbb{R}$  such that

$$X(\varphi, t, x) = \sum_{\alpha} c_{\alpha}(\varphi, t) \cdot H_{\alpha}(x),$$

where  $c_{\alpha}(\cdot, \cdot)$  is a mapping  $: \mathcal{S} \times \mathbb{R}^d \rightarrow \mathbb{R}$  for  $|\alpha| \geq 1$ , and for each  $\varphi \in \mathcal{S}$ , the mapping  $: \mathbb{R}^d \ni t \mapsto c_{\alpha}(\varphi, t)$  is Borel measurable, and if  $\alpha = 0$ ,  $c_0(\cdot)$  is just a measurable function on  $\mathbb{R}^d$ , independent of  $\varphi$ . Moreover,

$$\mathbb{E}[X(\varphi, t, \cdot)^2] = \sum_{\alpha} c_{\alpha}^2(\varphi, t) \cdot \alpha! < \infty$$

for any  $\varphi \in \mathcal{S}$ , and any  $t \in \mathbb{R}^d$ .

**Definition 6.** ( $(L^p)$ -Functional Process) We say  $X \in \mathcal{L}^p$  if  $X = X(\varphi, t, x) : \mathcal{S} \times \mathbb{R}^d \times \mathcal{S}' \rightarrow \mathbb{R}$  such that

- (a) a mapping  $: \mathbb{R}^d \ni t \mapsto X(\varphi, t, x)$  is Borel measurable for any  $\varphi \in \mathcal{S}$ ,  $\mu$ -a.e.  $x \in \mathcal{S}'$ ; and
- (b) a mapping  $: \mathcal{S}' \ni x \mapsto X(\varphi, t, x) \in (L^p)$  for any  $\varphi \in \mathcal{S}$ , any  $t \in \mathbb{R}^d$ .

The functional process  $X(\varphi, t, x)$  is called *positive* or a *positive noise* if  $X(\varphi, t, x) \geq 0$  holds  $\mu$ -a.e.  $x \in \mathcal{S}'$  for any  $\varphi \in \mathcal{S}$ , any  $t \in \mathbb{R}^d$ .

*Example 6.* (cf. [LOU]) Let  $X = X(\varphi, t, x)$ ,  $Y = Y(\varphi, t, x)$  be positive  $(L^2)$ -functional processes such that

$$X_{\varphi}(x) = \sum_{\alpha} a_{\alpha}(\varphi^{\otimes n}) \cdot H_{\alpha}(x), \quad Y_{\varphi}(x) = \sum_{\beta} b_{\beta}(\varphi^{\otimes n}) \cdot H_{\beta}(x).$$

Then the Wick product  $X \diamond Y$  is also positive.

**Theorem 11 [LOU].** (*Characterization of Positive Functional Process*) Let  $X \in (L^2)$ . Then  $X$  is positive ( $\mu$ -a.e.  $x \in \mathcal{S}'$ ) if and only if  $M^n(y) \equiv \tilde{X}^{(n)}(iy) \cdot \exp(-\frac{1}{2}|y|^2)$  is positive definite as a matrix of  $M(n \times n)$  for any  $n \in \mathbb{N}$ ,  $y \in \mathbb{R}^n$ , where  $\tilde{X}^{(n)}(z) \equiv \tilde{X}(z_1, z_2, \dots, z_n, 0, 0, \dots, 0)$ .

Let us consider the WN process. We shall introduce an interesting and important fact that the WN process provides a typical example of  $(L^p)$ -functional process, which very

often can be found useful in applications to stochastic partial differential equations[B], [D8], [HLOUZ1]. Set  $W(\varphi, t, x) \equiv W_{\varphi(t)}(x)$ , and define  $\varphi_t(u) = \varphi(t)(u) = \varphi(u - t)$ . Actually the WN process

$$W_{\varphi(t)}(x) = \langle x, \varphi_t \rangle = \int_{\mathbb{R}^d} \varphi_t(u) dB_u(x)$$

is naturally regarded as an  $(L^p)$ -functional process, i.e.  $W_{\varphi(t)} \in \mathcal{L}^p$ .

### 3.2 The Kondratiev Space Valued Process

**Definition 7.** (*Stochastic Distribution Valued Process*)

$$\Phi \equiv \Phi(t, p, \cdot) : \mathbb{R} \times \mathbb{R}^n \ni (t, p) \mapsto \Phi(t, p)(\cdot) \in (\mathcal{S})^{-1}$$

is regarded as a stochastic distribution valued process. We call such a function a  $(\mathcal{S})^{-1}$ -process.

Let us consider the derivative of  $(\mathcal{S})^{-1}$ -process. Let  $F(t)$  be a  $(\mathcal{S})^{-1}$ -process: namely,

$$F(t, \cdot) : \mathbb{R} \ni t \mapsto F(t, \cdot) \in (\mathcal{S})^{-1}.$$

**Definition 8.**  $\Xi \equiv \Xi(t_0) \in (\mathcal{S})^{-1}$  is said to be a derivative of  $(\mathcal{S})^{-1}$ -process  $F(t)$  with respect to  $t$  at  $t = t_0$  if there exists an element  $\Xi$  in  $(\mathcal{S})^{-1}$  such that

$$\frac{F(t_0 + h) - F(t_0)}{h} \rightarrow \Xi \quad \text{in } (\mathcal{S})^{-1} \quad (\text{as } h \rightarrow 0).$$

When the above limit exists, we write  $\Xi(t_0) \equiv \frac{dF}{dt}(t_0) (\in (\mathcal{S})^{-1})$ .

We set  $\mathcal{H}F(t) = \tilde{F}(t_0; z)$  and  $\mathcal{H}\Xi(t_0) = \tilde{\Xi}(t_0; z)$ . By virtue of the characterization of topology of  $(\mathcal{S})^{-1}$  (see Proposition 6 in §2.3), the aforementioned definition is equivalent to the following:

$$(17) \quad \frac{\tilde{F}(t_0 + h; z) - \tilde{F}(t_0; z)}{h} \rightarrow \tilde{\Xi}(t_0; z) \quad \text{as } h \rightarrow 0$$

holds pointwise, boundedly for any  $z \in \mathbb{B}_q(\delta)$  ( $\exists q < \infty, \delta > 0$ ). If the mapping  $t \mapsto \frac{d}{dt}\tilde{F}(t; z) = \frac{d}{dt}\mathcal{H}F(t)$  is continuous in  $t$ , and uniformly bounded for any  $z \in \mathbb{B}_q(\delta)$ , and any  $t$  in the neighborhood of  $t_0$ , then instead of the condition (17), the condition

$$(18) \quad \left\langle \frac{d}{dt}\tilde{F}(t; z) = \tilde{\Xi}(t; z) \quad \text{for } t = t_0, \text{ pointwise for each } z \in \mathbb{B}_q(\delta) \right\rangle$$

is just sufficient. Because, if Eq. (18) holds, we can write it as

$$\frac{\tilde{F}(t_0 + h; z) - \tilde{F}(t_0; z)}{h} = \frac{1}{h} \int_{t_0}^{t_0+h} \frac{d}{ds}\tilde{F}(s; z) ds \quad \text{for small } h,$$

and therefore, this expression turns out to be uniformly bounded for  $z \in \mathbb{B}_q(\delta)$  as  $h$  tends toward zero. If  $\frac{d}{dt}F$  exists and is  $t$ -continuous, then it follows that  $(\mathcal{S})^{-1}$ -process  $F(t) \in C^1$ .

## 4. The Stochastic Boundary Value Problem

### 4.1 Formulation

We consider the following stochastic boundary value problem:

$$(19) \quad \begin{aligned} du(t, r) &= \{\Delta u(t, r) + R(u(t, r))\}dt + h(t, r)u(t, r)dB_t, \\ 0 \leq t \leq T, \quad r &\in [0, 1], \\ u(t, 0) &= u(t, 1), \quad u(0, r) = u_0(r), \end{aligned}$$

where  $\Delta$  is the Laplacian and  $R(y)$  is a polynomial of  $y \in \mathbb{R}$ .  $B_t$  denotes a one dimensional Brownian motion.  $h, u_0$  are non random functions being continuous. In addition, assume  $u_0 \in C^3$ .

**Definition 9.** (Functional Process Solution)  $u \equiv u(\varphi, t, r, x)$  is said to be a  $(\mathcal{S})^{-1}$  functional process solution of Eq. (19) if

$$u : C_0^\infty(\mathbb{R}) \times [0, T] \times \mathbb{R} \rightarrow (\mathcal{S})^{-1}$$

is a Kondratiev space valued functional process and satisfies

$$(20) \quad u(t) = u_0(r) + \int_0^t \Delta_r u(s) ds + \int_0^t R^\diamond(u(s)) ds + \int_0^t h(s, r)u(s) \diamond W_{\varphi(s)}(x) ds,$$

for  $\varphi \in C_0^\infty(\mathbb{R})$  such that  $\varphi_s(t) = \varphi(t - s)$  with boundary condition.

We resort to the asymptotic solution theory. We shall say that  $u_k$  is an asymptotic solution for the problem (20) if there exists  $u_k = u_k(t, r)$  solving the reduced, modified or simplified equation, satisfying

$$(21) \quad u_k(t, r) \rightarrow u(t, r) \quad \text{in } (\mathcal{S})^{-1}.$$

Let  $u_k = u_k(\varphi, t, r, x, \omega)$  satisfies the following stochastic partial differential equation (SPDE for short):

$$(22) \quad \begin{aligned} u_k(t) &= u_{0k}(r) + \int_0^t \Delta_k u_k(s) ds + \int_0^t R^\diamond(u_k(s)) ds \\ &\quad + \int_0^t h_k(s, r)u_k(s) \diamond W_{\varphi(s)}(x) ds + M_k(t, r, \omega), \end{aligned}$$

with boundary condition, where  $\omega$  is an element of some proper probability space on which a martingale  $M_k$  is realized. We propose that the asymptotic problem for our case is to show that

$$\sup_t \|X_k(t) - \tilde{u}(t)\|_\infty \rightarrow 0 \quad (k \rightarrow \infty),$$

for  $T > 0$ , if we take Eq.(21) into consideration with characterization of topology in  $(\mathcal{S})^{-1}$  in accordance with Holden-Lindstrøm-Øksendal-Ubøe-Zhang formalism (cf. Proposition 6 in §2; see also [HLOUZ1], [HLOUZ2]).

$\tilde{u}$  is a solution solving

$$(23) \quad \frac{\partial}{\partial t} \tilde{u}(t) = (\Delta_r + c(t, r))\tilde{u}(t) + R(\tilde{u}(t)),$$

with the initial and boundary conditions, where we put  $c = h \cdot \tilde{W}_\varphi$ . The corresponding model for asymptotic solution is described as

$$(24) \quad dX_k(t) = (\Delta_k + c_k)X_k(t)dt + R(X_k(t))dt + dM_k(t),$$

with  $X_k(t, 0) = X_k(t, 1), \quad X_k(0, r) = u_{0k}(r).$

If we assume boundedness for  $R$  and the initial value, then the problem (23) has a continuous bounded solution by virtue of the implicit approximation scheme. Under further assumptions on  $R$  there exists a unique solution  $X_k$  for the problem (24). In fact we can construct it by employing the classical probability theory related to some jump type Markov processes with suitable conditions.

**Theorem 12.** *Under the assumption of convergence  $\|X_k(0) - \tilde{u}(0)\|_\infty \rightarrow 0$  in probability, then we get*

$$(25) \quad \lim_{k \rightarrow 0} \mathbb{P} \left( \sup_t \|X_k(t) - \tilde{u}(t)\|_\infty > \varepsilon \right) = 0,$$

as far as  $z \in \mathbb{B}_q(\delta)$ , for some positive  $\delta, q$ .

#### 4.2 The Probabilistic Model

Let us consider the totality of real valued step functions on  $[0, 1]$ , and we extend those functions periodically with period 1. We denote the extension by  $H_k$ . For  $f \in H_k$ , we define

$$\Delta_k f(r) = k^2 \left\{ f \left( r + \frac{1}{k} \right) - 2f(r) + f \left( r - \frac{1}{k} \right) \right\}.$$

We shall now introduce the discretized problem of Eq.(23), i.e.,

$$(26) \quad \frac{\partial}{\partial t} \tilde{u}_k(t, r) = (\Delta_k + c_k)\tilde{u}_k(t, r) + R(\tilde{u}_k(t, r)),$$

with the corresponding initial and boundary conditions. Then we have the bounded solution  $\tilde{u}_k(t)$  for all  $t$ , and

$$\sup_{t \in [0, T]} \|\tilde{u}_k(t) - \tilde{u}(t)\|_\infty \leq C(T, R, u_0) \cdot C'(k) \quad \text{for } T > 0,$$

with  $C'(k) = O(k^{-1})$ , ( $k \rightarrow \infty$ ).

While we consider the following SPDE driven by a martingale term  $M$ :

$$(27) \quad dX(t, r) = \{\Delta_r + c(t, r)\}X(t, r)dt + R(X(t, r))dt + dM_t.$$

We follow the standard notation in stochastic analysis (e.g. [IW]). Let  $M$  be a continuous square integrable local martingale on  $(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{F}_t)$ . If the quadratic variation process of  $M$  is given by an integral of  $G(s, \omega)^2$  relative to  $s$  over  $[0, t]$  where  $G(\neq 0)$  is a  $(\mathcal{F}_t)$ -predictable process and belongs to  $L^2([0, T])$  with probability one, then the representation theorem for martingales (p.90, [IW]) guarantees that there exists an extension  $(\Omega', \mathcal{F}', \mathbb{P}')$  with  $\mathcal{F}'_t$  and there exists an  $(\mathcal{F}'_t)$ -Brownian motion such that  $M(t) = \int_0^t G(s) dB(s)$ . So we assume that Eq.(27) has a solution  $(X, B)$  on  $(\Omega', \mathcal{F}', \mathbb{P}')$ . Define an  $\mathcal{A}^{1,1}$  process  $\gamma(t, X) = -c(t, r) X(t, r) G(t)^{-1}$ . Further suppose that

$$(28) \quad \mathbb{E} \exp \left( \frac{1}{2} \int_0^t |\gamma(s, X)|^2 ds \right) < \infty, \quad \forall t > 0,$$

$$(29) \quad \Gamma \exp \left\{ \int_0^t \gamma(s, X) dB(s) - \frac{1}{2} \int_0^t |\gamma(s, X)|^2 ds \right\} \text{ is a } (\mathcal{F}'_t) - \text{martingale.}$$

Put  $\hat{\mathbb{P}} = \Gamma \cdot \mathbb{P}'$  and  $\hat{B}(t) = B(t) - \int_0^t \gamma(s, X) ds$ . An application of the Girsanov theorem [G] allows that  $\hat{B}(t)$  becomes a  $(\mathcal{F}'_t)$ -Brownian motion on  $(\Omega', \mathcal{F}', \hat{\mathbb{P}})$ . Therefore  $(X, \hat{B})$  on  $(\Omega', \mathcal{F}', \hat{\mathbb{P}})$  solves the stochastic equation:

$$(30) \quad dX(t, r) = \Delta_r X(t, r)dt + R(X(t, r))dt + d\hat{M}_t,$$

with  $\hat{M}(t) = \int_0^t G(s) d\hat{B}(s)$ . On the other hand, we consider the stochastic process  $U(t)$  describing a density dependent birth and death process. In fact, let  $U(t) = (U_1(t), \dots, U_k(t))$  be a  $\mathbb{N}^k$ -valued jump type Markov process whose Markovian particle may diffuse on the circle in accordance with simple random walk with jump rate  $2k^2$ , and besides with birth rate  $pR_1(U_i/p)$  and with death rate  $pR_2(U_i/p)$  where  $p$  is a given parameter and  $R = R_1 - R_2$ . We can construct such a process  $U(t)$  by classical probability theory and realize it as a cadlag process on some suitable probability space.  $\mathcal{F}_t^p$  denotes the completed  $\sigma$ -field of  $\sigma(U(s); s \leq t)$ . Let  $T(\omega)$  be an  $\mathcal{F}_t^p$  stopping time satisfying

$$\{\omega \in \Omega; T(\omega) \leq t\} \in \mathcal{F}_t^p \text{ for } \forall t, \quad \text{and} \quad \sup_t \{U(t \wedge T(\omega)) \cdot I_{T(\omega) > 0}(\omega)\} < \infty.$$

Then by martingale theory [LS] it follows that

$$U_i(t \wedge T(\omega)) - \int_0^{t \wedge T(\omega)} \Phi(U, R, p, i; s) ds$$



is an  $\mathcal{F}_t^p$ -martingale [BL], where we set  $\Phi(U, R, p, i; s) = pR(U_i(s)/p) + k^2\{U_{i+1}(s) + U_{i-1}(s) - 2U_i(s)\}$ . Define

$$X_k(t, r) := U_i(t)/p \quad \text{for } r \in [i/k, (i+1)/k), \quad i = 1, 2, \dots, k-1.$$

Thus we attain that the  $H_k$  valued Markov process  $X_k$  satisfies the discretized version of Eq.(30):

$$(31) \quad dX_k(t, r) = \Delta_k X_k(t, r)dt + R(X_k(t, r))dt + d\hat{M}_k(t).$$

#### 4.3 Law of Large Numbers for the Stochastic Problem

In order to prove Eq.(25) it is sufficient to show that

$$\mathbb{P}\left\{\sup_t \|X_k(t) - \tilde{u}_k(t)\|_\infty > \varepsilon\right\}$$

converges to zero as  $k$  tends toward infinity. Set  $T_t = \exp(t\Delta_k)$  and

$$Y_k(t) = \int_0^t T_{t-s} d\hat{M}_k(s \wedge T(\omega)).$$

Moreover, a simple calculation leads to  $\|\delta X_k(t \wedge T(\omega))\|_\infty = O(p^{-1})$  with precise estimates. On this account, the problem can be attributed finally to computation of the term  $\sup_t \|Y_k(t)\|_\infty$ . In fact we need to estimate

$$\sup_{t \in [a, b]} \|Y_k(t)\|_\infty \leq C_1 \|Y_k(a)\|_\infty + C_2 \sup_{t \in [a, b]} \|M_k(t \wedge T(\omega)) - M_k(a \wedge T(\omega))\|_\infty.$$

By making use of Gronwall's inequality, Markov' inequality and Doob's inequality, we deduce that

$$\mathbb{P}\left\{C_3(T) \sup_{t \in [c, d]} \|Y_k(t)\|_\infty > \varepsilon\right\} \leq C_4(k, p, \varepsilon),$$

because we applied martingale theory. For the final estimate, we need Lemma 4.4, p.135 [BL].

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